J. Fluid Mech. (2001), *vol.* 443, *pp.* 231–236. Printed in the United Kingdom

An exact form of Lilley's equation with a velocity quadrupole/temperature dipole source term

By M. E. GOLDSTEIN

National Aeronautics and Space Administration, Glenn Research Center, Cleveland, Ohio 44135, USA

(Received 15 May 2001 and in revised form 14 June 2001)

There have been several attempts to introduce approximations into the exact form of Lilley's equation in order to express the source term as the sum of a quadrupole whose strength is quadratic in the fluctuating velocities and a dipole whose strength is proportional to the temperature fluctuations. The purpose of this note is to show that it is possible to choose the dependent (i.e. the pressure) variable so that this type of result can be derived directly from the Euler equations without introducing any additional approximations.

1. Introduction

The subject of aeroacoustics was first put on a rational basis by Lighthill (1952, 1954) when he rearranged the Navier–Stokes (Euler) equations into the form of a linear wave equation for a medium at rest with a quadrupole-type source term. This source includes a contribution from a pressure/density term, which Lighthill attributed to non-isentropic fluctuations in the flow. However, Lilley (1974, 1996) argued that isentropic fluctuations in a heated jet can actually produce a dipole source. Comparison with experiment showed that this description yielded the correct sound pressure levels at all Mach numbers and jet temperature ratios. This suggests that Lilly's result is real and not just a consequence of the theory. It implies, among other things, that the temperature effect changes sign at a jet Mach number of 0.8.

In any event, the crucial step in Lighthill's so-called acoustic analogy approach amounts to assuming that the source term is in some sense known or that it can at least be modelled in some approximate fashion. While this approach was remarkably successful in predicting the gross features of the sound radiation from turbulent air jets, the commercial aircraft industry soon realized that it needed a much more sensitive tool with the capability of predicting how even relatively small changes in the flow would affect the radiated sound. This motivated generations of researchers to seek improvements in the Lighthill approach. Early efforts were focused on accounting for mean flow interaction effects and there were a number of attempts to accomplish this by applying *ad hoc* corrections to the original Lighthill predictions. A more satisfying approach was the one adopted by Phillips (1960), Lilley (1974) and others, which amounted to deriving inhomogeneous moving media wave equations for the sound generation process.

The dominant part of the Lighthill source term is quadratic in the total flow velocity, which can be decomposed into a mean plus a fluctuating component. The source function therefore contains terms that are both linear and quadratic in the fluctuating velocity components. Lilley (1974) argued that the linear terms, which are typically much larger than the quadratic quantities, do not actually radiate any sound and should, therefore, not be included in the source function, since they would tend to dominate over the much smaller quadratic terms which are the true sources of sound. Including them would cause the sound source to be contaminated by the small but inevitable errors resulting from the actual computation of these terms and would thereby lead to inaccurate predictions of the sound field.

However, it turns out that the equation derived by Lilley has a complicated source term (Colonius, Lele & Moin 1997), which is not of the physically expected form, i.e. the sum of a quadrupole whose strength is quadratic in the fluctuating velocities and a dipole whose strength is proportional to the temperature fluctuations. There have been a number of attempts to obtain such a source term by introducing various approximations into Lilley's equation. The purpose of this note is to show that a source of this type can be obtained by making an appropriate choice of the dependent (i.e. the pressure) variable.

2. The Lilley equation and related background information

Lilley (1974) showed that for an ideal gas the Navier–Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\rho v_j = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}\rho v_i + \frac{\partial}{\partial x_j}\rho v_i v_j + \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j} e_{ij}, \tag{2.2}$$

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = e_{ij}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}, \tag{2.3}$$

where

$$s = c_p \ln(p^{1/\kappa}/\rho) \tag{2.4}$$

denotes the entropy, c_p denotes the specific heat at constant pressure, $\kappa \equiv c_p/c_v$ denotes the specific heat ratio, t denotes the time, $\boldsymbol{x} \equiv \{x_1, x_2, x_3\}$ are Cartesian coordinates, p denotes the pressure, ρ the density, $\boldsymbol{v} = \{v_1, v_2, v_3\}$ the fluid velocity, e_{ij} the viscous stress tensor, q_i the heat flux vector and

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}$$

is the convective derivative, can be rearranged into the third-order wave equation (see Goldstein 1976, p. 253)

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\mathrm{D}^2 \Pi}{\mathrm{D}t^2} - \frac{\partial}{\partial x_i} c^2 \frac{\partial \Pi}{\partial x_i}\right) + 2\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j} c^2 \frac{\partial \Pi}{\partial x_i} = -\frac{\partial v_j}{\partial x_i}\frac{\partial v_k}{\partial x_j}\frac{\partial v_i}{\partial x_k} + \Psi, \tag{2.5}$$

where

$$\Pi \equiv \frac{1}{\kappa}\ln\frac{p}{p_o}, \tag{2.6}$$

$$c^2 = \kappa R T = \kappa\, p/\rho. \tag{2.7}$$

is the squared sound speed, R is the gas constant, T is the temperature. Ψ represents the effects of entropy fluctuations and fluid viscosity, which are generally considered to be unimportant and are therefore neglected in much of the following discussion.

The most general unidirectional transversely sheared mean flow that satisfies the Euler equations is given by

$$v_i = \delta_{i1} U(x_2, x_3), \quad p = p_o = \text{constant}, \quad T = T_o(x_2, x_3). \tag{2.8}$$

Subtracting this from the actual velocity and thermodynamic variables and moving terms that are nonlinear in the resulting deviations to the right-hand side of (2.5) leads to the inhomogeneous Pridmore-Brown (1957) equation

$$\bar{L}_o \Pi = \Gamma, \tag{2.9}$$

where

$$\bar{L}_o \equiv \frac{\mathrm{D}_o}{\mathrm{D}t}\left(\frac{\mathrm{D}_o^2}{\mathrm{D}t^2} - \frac{\partial}{\partial x_i}\overline{c^2}\frac{\partial}{\partial x_i}\right) + 2\frac{\partial U}{\partial x_i}\frac{\partial}{\partial x_1}\overline{c^2}\frac{\partial}{\partial x_i} \tag{2.10}$$

is the Pridmore-Brown operator,

$$\frac{\mathrm{D}_o}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x_1}, \tag{2.11}$$

is the convective derivative based on the mean flow velocity and

$$\overline{c^2} = \kappa R T_o. \tag{2.12}$$

Notice that

$$\Pi = \frac{1}{\kappa}\ln\left(1 + \frac{p'}{p_o}\right) \simeq \frac{1}{\kappa}\frac{p'}{p_o}, \tag{2.13}$$

when $p' \equiv p - p_o \ll p_o$, i.e. when the pressure fluctuations are small.

The detailed expression for Γ is given in Colonius *et al.* (1997). This result is still exact but the source term is now very complicated and even more importantly does not exhibit the quadrupole/dipole form originally proposed by Lighthill (1952) and Lilley (1974). Lighthill emphasized the importance of properly exhibiting the correct multipole order of the source term before introducing specific modelling assumptions for this quantity and Colonius *et al.* (1997) showed the extreme sensitivity of the predicted sound field to the detailed assumptions about the form of the source.

Goldstein (1984) carried out a systematic second-order asymptotic expansion and introduced a new dependent variable to show that

$$\bar{L}_o \pi' = \frac{\mathrm{D}_o}{\mathrm{D}t}\frac{\partial f_i'}{\partial x_i} - 2\frac{\partial U}{\partial x_j}\frac{\partial f_j'}{\partial x_1} \tag{2.14}$$

to within second-order accuracy. $\bar{L}_o$ is defined in (2.10) and the new dependent variable π' is defined by

$$\pi' \equiv \Pi + \tfrac{1}{2}\Pi^2. \tag{2.15}$$

Also,

$$f_i' \equiv -\frac{\partial}{\partial x_j} v_i' v_j' - (c^2)'\frac{\partial \Pi}{\partial x_i}, \tag{2.16}$$

$$v_i' \equiv v_i - \delta_{i1} U(x_2, x_3), \tag{2.17}$$

and

$$(c^2)' \equiv \kappa R(T - T_o) \tag{2.18}$$

is the fluctuating sound speed (notice that the definition of Π differs from the one used in Goldstein 1984).

The source term in this equation is identical to the one that would be produced by an externally applied force $\boldsymbol{f}' = \{f_1', f_2', f_3'\}$, and is therefore properly interpreted as a dipole. The first term in $\boldsymbol{f}'$ represents the source that would be produced by the fluctuating shear stress $v_i'v_j'$ and can therefore be interpreted as a quadrupole. The remaining term is a dipole source produced by the temperature fluctuations

$$T' \equiv T - T_o. \tag{2.19}$$

The quadrupole source scales like v'^2/l, where l is a characteristic length of the turbulence and the second term scales like $(U/c_0)^2(v'^2/l)$, and should therefore be negligible compared to the first when the Mach number is small (Morfey, Szewczyk & Tester 1978).

Colonius *et al.* (1997) showed that they could accurately reproduce the numerically predicted sound field radiated from a low-Mach-number shear layer by substituting the numerically computed values for

$$f_i' \approx -\frac{\partial}{\partial x_j} v_i'v_j' \tag{2.20}$$

and U into (2.14) and numerically solving the resulting linear equation for π'. However, the Goldstein expansion, on which this result is based, is, at best, only locally valid, since nonlinear effects eventually dominate the near-field disturbances and cause the expansion to break down. And since the acoustic field depends on the global solution to the problem, this approach does not lead to a rigorous derivation of the basic acoustic analogy equation. Lilley (1999, 2000) recently proposed an exact inhomogeneous Pridmore-Brown equation with a quadrapole type source term for the special case of a constant shear–constant sound speed base flow.

3. The exact equation

The purpose of this note is to show that it is possible to obtain an exact rearrangement of the Navier–Stokes (Euler) equations that leads to a third-order convective wave equation with a simple source term that consists of a velocity quadrupole plus a fluctuating temperature dipole by introducing an appropriate dependent variable to represent the pressure fluctuations.

To this end, we neglect viscous and heat conduction effects (we indicate below how the final result can be modified to include these effects by adding an addition term to the source function) and substitute (2.4) into (2.3) and (2.7) to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}(p^{1/\kappa}/\rho) = 0. \tag{3.1}$$

Then multiplying (2.1) and (2.2) by $p^{1/\kappa}/\rho$, differentiating by parts and using (2.7) and (3.1) shows that

$$\frac{\partial}{\partial t}p^{1/\kappa} + \frac{\partial}{\partial x_j}(p^{1/\kappa}v_j) = 0, \tag{3.2}$$

$$\frac{\partial}{\partial t}p^{1/\kappa}v_i + \frac{\partial}{\partial x_j}(p^{1/\kappa}v_iv_j) + c^2\frac{\partial p^{1/\kappa}}{\partial x_i} = 0, \tag{3.3}$$

which, upon introducing (2.11) and (2.17), can be written as

$$\frac{\mathrm{D}_o}{\mathrm{D}t}p^{1/\kappa} + \frac{\partial}{\partial x_j}(p^{1/\kappa}v_j') = 0 \tag{3.4}$$

and

$$\frac{\mathrm{D}_o}{\mathrm{D}t}(p^{1/\kappa}v_i') + \delta_{1i}p^{1/\kappa}v_j'\frac{\partial U}{\partial x_j} + c^2\frac{\partial p^{1/\kappa}}{\partial x_i} = -\frac{\partial}{\partial x_j}(p^{1/\kappa}v_i'v_j'), \tag{3.5}$$

where we have used (3.4) to simplify (3.5). Upon introducing the new dependent variables

$$u_i \equiv (p/p_o)^{1/\kappa}v_i', \tag{3.6}$$

$$\pi \equiv (p/p_o)^{1/\kappa} - 1, \tag{3.7}$$

and using (2.12) and (2.18), these become the inhomogeneous 'linearized Euler equations'

$$\frac{\mathrm{D}_o\pi}{\mathrm{D}t} + \frac{\partial u_j}{\partial x_j} = 0, \tag{3.8}$$

$$\frac{\mathrm{D}_o u_i}{\mathrm{D}t} + \delta_{1i}\boldsymbol{u}\cdot\nabla U + \overline{c^2}\frac{\partial \pi}{\partial x_i} = f_i, \tag{3.9}$$

where the externally applied force f_i is now given by

$$f_i = -\frac{\partial}{\partial x_j}(1+\pi)v_i'v_j' - (c^2)'\frac{\partial \pi}{\partial x_i}. \tag{3.10}$$

It is worth noting that the, now conventional, terminology 'linearized Euler equations' is somewhat misleading because (3.8) and (3.9) are, in fact, nonlinear, since the nonlinear inhomogeneous terms are *a priori* unknown. In the more general viscous case a monopole source term $(p^{1/k}/c_p)\mathrm{D}s/\mathrm{D}t$ will appear on the right-hand side of (3.8) and f_i will contain the additional term $(p^{1/k}/c_p)v_i\mathrm{D}s/\mathrm{D}t + (p^{1/k}/\rho)\partial e_{ij}/\partial x_j$.

Equations (3.8) and (3.9) are identical in form to the linearized equations discussed in Chapter 1 of Goldstein (1976), where it is shown (by taking the convective derivative of the first equation and the divergence of the second, subtracting the results and then using the second equation with $i = 1$ to eliminate the velocity fluctuation on the left-hand side) that they can be rearranged into the inhomogeneous Pridmore-Brown equation

$$\bar{L}_o\pi = \frac{\mathrm{D}_o}{\mathrm{D}t}\frac{\partial f_i}{\partial x_i} - 2\frac{\partial U}{\partial x_j}\frac{\partial f_j}{\partial x_1}, \tag{3.11}$$

which is identical to (2.14) but with the pressure fluctuation π now given by (3.7) and the externally applied force $\boldsymbol{f}$ now given by (3.10) rather than by (2.16). Notice that

$$\pi \rightarrow \frac{1}{\kappa}p'/p_0 \quad \text{when} \quad p' \equiv p - p_0 \rightarrow 0. \tag{3.12}$$

Finally, it is worth noting that (3.11) can be written as

$$L_o\pi = \frac{\mathrm{D}_o}{\mathrm{D}t}\frac{\partial \tilde{f}_i}{\partial x_i} - 2\frac{\partial U}{\partial x_i}\frac{\partial \tilde{f}_i}{\partial x_1}, \tag{3.13}$$

where L_o is the same as (2.10) but with $\overline{c^2}$ replaced by $c^2 = \overline{c^2} + (c^2)'$ and

$$\tilde{f}_i \equiv -\frac{\partial}{\partial x_j}(1+\pi)v_i'v_j' \tag{3.14}$$

is now a pure quadrupole source.

Aside from the definition of the pressure fluctuation, the only difference between (2.14) and (3.11) is the appearance of the pressure fluctuation factor $(1+\pi)$ in the quadrupole strength $(1+\pi)v_i'v_j'$. Since π should be of the order of the turbulence intensity squared (which is typically small compared to unity) and since substituting exact values of U and f_i into (3.11) should yield exactly the same result as the direct numerical solution for the sound field, this explains why Colonius *et al.* (1997) were able to obtain such good agreement using the two different approaches.

Lighthill indicated that the basis of his acoustic analogy is the demonstration that there is an exact analogy between the density fluctuations in any real flow and those produced by a quadrupole source (or, as subsequently shown by Lilley 1974, 1996, a dipole plus a quadrupole source) in an ideal stationary acoustic medium. The present result shows that, aside from viscous and heat conduction effects, there is an exact analogy between the $(p/p_o)^{1/k}$ fluctuations in any real flow and the corresponding linear fluctuations in this quantity produced by a quadrupole plus a temperature dipole source in an arbitrary ideal transversely sheared mean flow. It is important to note that this does not imply that (3.11), or for that matter any other 'acoustic analogy' equation, can provide an unambiguous identification of the sources. This result is only useful when the base flow bears some resemblance to the actual fluid motion and, even then, can only serve as a guide for identifying and ultimately modelling the true sources of sound.

The author wishes to thank Professor Jeoffrey Lilley for his constructive comments on the manuscript and Dr Stewart Leib for commenting on the draft.

REFERENCES

Colonius, T., Lele, S. K. & Moin, P. 1997 Sound generation in a mixing layer. *J. Fluid Mech.* **330**, 375–409.

Goldstein, M. E. 1976 *Aeroacoustics.* McGraw-Hill.

Goldstein, M. E. 1984 Aeroacoustics of turbulent shear flows. *Ann. Rev. Fluid Mech.* **16**, 263–295.

Lighthill, M. J. 1952 On sound generated aerodynamically: I. General theory. *Proc. R. Soc. Lond.* A **211**, 564–587.

Lighthill, M. J. 1954 On sound generated aerodynamically: II. Turbulence as a source of sound. *Proc. R. Soc. Lond.* A **222**, 1–32.

Lilley, G. M. 1974 On the noise from jets, noise mechanisms. *AGARD–CP*–131, pp. 13.1–13.12.

Lilley, G. M. 1996 The radiated noise from isotropic turbulence with applications to the theory of jet noise. *J. Sound. Vib.* **190**, 463–476.

Lilley, G. M. 1999 On the refraction of jet noise. *Proc. 6th Intl Congr. on Sound and Vibration, Copenhagen, Denmark*, pp. 3581–3588.

Lilley, G. M. 2000 The acoustic analogy review. *Proc. Jet Noise Workshop (sponsored by the Aeroacoustics Research Consortium) Cleveland, OH* (to be published).

Morfey, C. L., Szewczyk, V. M. & Tester, B. J. 1978 New scaling laws for hot and cold jet mixing noise based on a geometrics acoustics model. *J. Sound Vib.* **61**, 255–292.

Phillips, O. M. 1960 On the generation of sound by supersonic turbulent shear layers. *J. Fluid Mech.* **9**, 1–28.

Pridmore-Brown, D. C. 1958 Sound propagation in a fluid flowing through an attenuating duct. *J. Fluid Mech.* **4**, 393–406.